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Two-state spectral-free solutions of the Frenkel–Moore simplex equation

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Abstract. Whilst many solutions have been found for the quantum Yang–Baxter equation (QYBE), there are fewer known solutions available for its higher dimensional generalizations: Zamolodchikov's tetrahedron equation (ZTE) and Frenkel and Moore's simplex equation (FME). In this paper, we present families of solutions to the FME which may help us to understand more about higher dimensional generalization of the QYBE.

1. Introduction

The quantum Yang–Baxter equation (QYBE) plays a pivotal role in the study of two-dimensional integrable models, quantum groups, conformal field theory and the study of link polynomials in knot theory. A systematic study of the solutions of the QYBE shows that there are an infinite number of two-dimensional exactly solvable models in classical statistical mechanics [1, 2].

Using a computer algebra method, Hietarinta [3, 4] has obtained the complete classification of all two-state solutions of the QYBE. He has also extended his work in search of three-state solutions of the constant YBE [5] by studying all upper triangular ansatze.

Higher dimensional generalization of the QYBE is possible. By considering the scattering amplitudes of straight strings in $2+1$ dimensions, Zamolodchikov [6, 7] derives a three-dimensional equivalent of QYBE, commonly called the tetrahedron equation (ZTE):

$$\begin{aligned} &R_{123}(\theta_1,\theta_2,\theta_3)R_{145}(\theta_1,\theta_4,\theta_5)R_{246}(\theta_2,\theta_4,\theta_6)R_{356}(\theta_3,\theta_5,\theta_6) \\ &\qquad = R_{356}(\theta_3,\theta_5,\theta_6)R_{246}(\theta_2,\theta_4,\theta_6)R_{145}(\theta_1,\theta_4,\theta_5)R_{123}(\theta_1,\theta_2,\theta_3) \end{aligned} \tag{1}$$

where $R_{123} = R\otimes \mathbb{1}$ etc and $R \in \mathrm{End}(V\otimes V\otimes V)$ for some vector space V. In the same paper, he also ingeniously provided a non-trivial spectral-dependent two-state solution. Baxter [8] subsequently proved that the solution conjectured in Zamolodchikov's paper satisfies the tetrahedron equation. The tetrahedron equation is by no means simple. Even in the two-state case, there are 2^{14} consistency equations with 2^6 variables. An N-state generalization of the tetrahedron solution has also been found using a free-fermion model on the three-dimensional lattice [9, 10].

The tetrahedron equation is not the only higher dimensional generalization. By investigating the symmetry inherent in QYBE, Frenkel and Moore [11] suggested another possible generalization (FME):

$$R_{123}R_{124}R_{134}R_{234} = R_{234}R_{134}R_{124}R_{123} \tag{2}$$

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where $R_{123} = R \otimes \mathbb{I}$ etc and $R \in \mathrm{End}(V \otimes V \otimes V)$ for some vector space V. While much work [9, 10, 12–16] has been done to relate Zamolodchikov's tetrahedron equation to three-dimensional lattice models, less effort [17, 18] has so far been directed at the Frenkel and Moore generalization.

In section 2, we describe some known symmetries associated with FME, and review other works done on FME. We also show that, unlike ZTE, a cross-diagonal ansatz always satisfies two-state FME. In section 3, we briefly describe the technique in our work and then present our results in section 4.

2. Frenkel–Moore simplex equation

2.1. General symmetries

There are some significant differences between Zamolodchikov's tetrahedron equation and Frenkel and Moore's simplex equation. Essentially, the underlying vector spaces on which the operator R acts differ. Further, the operator R in Zamolodchikov's tetrahedron equation seems to be local whilst the operator R in Frenkel and Moore's version possesses a global labelling scheme [11, 19].

By choosing an appropriate basis for V, Frenkel and Moore's equation becomes

$$R_{abc}^{i_1i_2i_3} R_{i_1i_2d}^{i_4i_5i_6} R_{i_4i_3i_6}^{ei_7i_8} R_{i_5i_7i_8}^{fgh} = R_{bcd}^{i_1i_2i_3} R_{ai_2i_3}^{i_4i_5i_6} R_{i_4i_1i_6}^{i_7i_8h} R_{i_7i_8i_5}^{efg} \tag{3}$$

where repeated indices are summed over.

The equation does not possess a spectral parameter but it is invariant under a similarity transformation [4, 17] similar to the QYBE:

$$R \to \kappa(Q \otimes Q \otimes Q)R(Q^{-1} \otimes Q^{-1} \otimes Q^{-1}) \tag{4}$$

for some non-singular matrix $Q \in \mathrm{End}(V)$. It is also invariant under permutation of the indices, namely

$$R_{ijk}^{lmn} \to R_{(i+r) \bmod d\ (j+r) \bmod d\ (k+r) \bmod d}^{(l+r) \bmod d\ (m+r) \bmod d\ (n+r) \bmod d} \tag{5a}$$

$$R_{ijk}^{lmn} \to R_{lmn}^{ijk} \tag{5b}$$

$$R_{ijk}^{lmn} \to R_{kji}^{nml} \tag{5c}$$

where $r = 1, 2, 3, \ldots, d-1$, and $\dim(V) = d$. This is known as discrete symmetry [4].

The symmetry transformations (5*b*) and (5*c*) imply the symmetry transformation:

$$R_{ijk}^{lmn} \to R_{nml}^{kji}. \tag{6}$$

Indeed, if we consider all possible transformations involving permutation of the indices $\{i, j, k, l, m, n\}$, (5*b*) and (5*c*) are the only ones that will allow the FME to remain invariant. In this paper, we only consider the case when $\dim(V) = 2$.

2.2. Other works on FME

By considering total symmetric ansatz, namely, R-matrices in which

$$R_{ijk}^{lmn} = R_{jik}^{mln} = R_{ikj}^{lnm} = R_{kji}^{nml} = R_{jki}^{mnl} = R_{kij}^{nlm} \tag{7}$$

the authors of [17] have successfully listed all totally symmetric solutions of FME. They have found five independent solutions after eliminating those solutions which are related to each other under symmetry transformations.

Zheng and Zhang [18] have also constructed some beautiful solutions of FME. They considered an ansatz of the form $\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$, where X and Z are solutions of QYBE related to the superalgebra, the Temperly–Lieb algebra and the Birman–Wenzl algebra.

2.3. Cross-diagonal ansatz

Any diagonal ansatz, which is just an R-matrix with only diagonal entries, will satisfy a simplex equation, be it ZTE or FME. However, the statement is not true if we consider cross-diagonal R-matrices. A cross-diagonal R-matrix is one in which the only non-zero elements are $R_{ijk}^{(i+1)\mathrm{mod}\ 2\ (j+1)\mathrm{mod}\ 2\ (k+1)\mathrm{mod}\ 2}$. Whilst this ansatz does not necessarily satisfy the tetrahedron equation unless certain conditions hold, it will always satisfy the two-state FME.

To see this fact, we simply consider $R_{ijk}^{\mu\nu\sigma} \neq 0$ provided $i = \bar{\mu}$, $j = \bar{\nu}$, $k = \bar{\sigma}$, where $\mu + \bar{\mu} = 1 (\mathrm{mod}\ 2)$, $\bar{\mu}$ being the complement of μ. Substituting into FME, we see that the only non-zero terms on the left- and right-hand side of the equation are

$$R_{abc}^{\bar{a}\bar{b}\bar{c}} R_{\bar{a}\bar{b}d}^{ab\bar{c}} R_{a\bar{c}\bar{d}}^{\bar{a}cd} R_{bcd}^{\bar{b}\bar{c}\bar{d}}. \tag{8}$$

In contrast, if we substitute this form of the R-matrix into the ZTE, the left- and right-hand side of the equation do not necessarily cancel, and we require the terms

$$R_{abc}^{\bar{a}\bar{b}\bar{c}} R_{\bar{a}de}^{a\bar{d}\bar{e}} R_{\bar{b}\bar{d}f}^{bd\bar{f}} R_{\bar{c}\bar{e}\bar{f}}^{cef} - R_{cef}^{\bar{c}\bar{e}\bar{f}} R_{bd\bar{f}}^{\bar{b}\bar{d}f} R_{a\bar{d}\bar{e}}^{\bar{a}de} R_{\bar{a}\bar{b}\bar{c}}^{abc} \tag{9}$$

to be zero. One such possibility corresponds to equation (13) in Hietarinta's paper [4], which is

$$R_{111}^{222} = R_{222}^{111} = a \qquad R_{112}^{221} = R_{212}^{121} = b \qquad R_{121}^{212} = R_{212}^{121} = c \qquad R_{122}^{211} = R_{211}^{122} = d \tag{10}$$

where a, b, c and d are some arbitrary parameters with all other entries of the R-matrix being zero.

3. Technique

In this paper, most of the algebraic computations have been done using the computer algebra, MATHEMATICA [20]. The method used is similar to that employed by Hietarinta in his analysis of the QYBE [3]. Using a short program, we churn out all 256 equations from the FME, using a suitably chosen ansatz. We then analyse the 256 equations for the unknowns. These equations are generally trivial, though in the simplest case of a diagonal ansatz with one off-diagonal element, there can be as many as seven different 'quartic' equations with nine unknowns.

The ansatz that we choose initially are basically diagonal ones or cross-diagonal ones with an increasing number of off-diagonal elements. By systematically increasing the number of such off-diagonal elements, we hope to push the list as far as possible. We only manage to exhaust all listing up till two off-diagonal elements. The task gets very involved as the number of off-diagonal elements increases to three. In the case of two off-diagonal elements, there are still 1540 cases, although this number can be substantially reduced by looking at the discrete symmetries mentioned in section 2.1. In the case of three off-elements, for instance, there are altogether 27 720 possible cases which can be cut down easily by the symmetries in the indices of the equations.

4. Results

Solutions to the FME are not always independent. Due to invariance under the symmetry transformations, many solutions are related to each other and the number of different solutions can largely be reduced.

4.1. Solutions with only one non-zero off-diagonal element

There are 56 possible off-diagonal positions for the non-zero elements. However, if we consider all possible discrete symmetries, we need to consider only 12 different positions for the non-zero off-diagonal element. A convenient choice of these positions is shown in the array below:

$$\begin{pmatrix} \cdot & \cdot & s_1 & s_2 & s_3 & s_4 & \cdot & d_1 \\ \cdot & \cdot & s_5 & s_6 & s_7 & s_8 & d_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & d_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & d_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{11}$$

In addition, it is found that the cases corresponding to the non-zero element at the following positions s_7, s_8 and d_3 respectively do not yield non-singular solutions. Thus, we effectively have a total of nine different positions to consider for the non-zero-element.

To present our solutions more compactly, we shall write the solutions in the form

$$\{R^{111}_{111}, R^{112}_{112}, R^{121}_{121}, R^{122}_{122}, R^{211}_{211}, R^{212}_{212}, R^{221}_{221}, R^{222}_{222}\} \tag{12}$$

where R^{lmn}_{ijk} denotes the only non-zero off-diagonal element. Further, without any loss of generality, we shall set R^{111}_{111} to unity.

The solutions are as follows.

(i) s_1: $R^{121}_{111} = k \neq 0$.

$$\left\{1, 1, 1, a, 1, \frac{1}{a}, a, b\right\}. \tag{13}$$

There are also three other solutions in which complex entries occur:

$$\left\{1, -\omega, -1, a, -\omega^2, \frac{\omega^2}{a}; -\omega^2 a, b\right\} \tag{14}$$

where $\omega^3 = 1$.

(ii) s_2: $R^{122}_{111} = k \neq 0$.

$$\{1, \pm 1, \xi, \pm\xi^6, \pm\xi^6, \pm\xi, \xi^{12}, \pm\xi^2\} \tag{15}$$

$$\{1, \zeta, \pm\zeta, -1, \zeta^7, \pm\zeta, -\mathrm{i}, \zeta^3\} \tag{16}$$

where $\xi^7 = \pm 1$ and $\zeta^8 = \pm 1$.

(iii) s_3: $R^{211}_{111} = k \neq 0$.

$$\{1, a, b, c, -1, abd, d, e\} \tag{17}$$

where $b, d = \pm 1$.

(iv) s_4: $R^{212}_{111} = k \neq 0$.

$$\left\{1, a, b, \frac{1}{ab}, a, \pm\frac{a^2}{b}, \frac{1}{ab}, \pm a^2 b^2\right\} \tag{18}$$

$$\left\{1, a, \pm a^2, \pm\frac{1}{a^3}, -a, -1, \mp\frac{1}{b^3}, \mp b^6\right\} \tag{19}$$

$$\left\{1, a, \pm a^2, \pm\frac{1}{a^3}, \mathrm{i}a, -1, \mp\frac{\mathrm{i}}{b^3}, \mp b^6\right\} \tag{20}$$

$$\left\{1, a, \pm a^2, \pm\frac{1}{a^3}, -\mathrm{i}a, -1, \mp\frac{\mathrm{i}}{b^3}, \mp b^6\right\}. \tag{21}$$

(v) s_5: $R^{121}_{112} = k \neq 0$. This is a more complicated case than the rest and probably deserves more discussion. If we now write the solutions in the form:

$$\{R^{111}_{111}, R^{112}_{112}, R^{121}_{121}, R^{122}_{122}, R^{211}_{211}, R^{212}_{212}, R^{221}_{221}, R^{222}_{222}; R^{121}_{112}\} \tag{22}$$

the solution takes the form

$$\{1, a, b, c, b, c, c, d; k\} \tag{23}$$

where a, b, c and k are not independent, but related to each other by the equations

$$c^2 = ab + ck \tag{24a}$$

$$(a^2 - 1)b + k(a + b) = 0. \tag{24b}$$

Suppose we allow $b = c = 1$, we will get the solution:

$$\{1, a, 1, 1, 1, 1, d; 1 - a\}. \tag{25}$$

Other possibilities exist. If we allow $a = 1$ and $a = b$, we easily get

$$\left\{1, 1, -1, c, -1, c, c, d; \frac{1 + c^2}{c}\right\} \tag{26}$$

and

$$\left\{c \text{ or } \frac{b^2}{c}, b, b, c, b, c, c, d; \frac{c^2 - b^2}{c}\right\} \tag{27}$$

respectively.

(vi) s_6: $R^{122}_{112} = k \neq 0$.

$$\{1, 1, 1, \omega, \omega, a, 1, \omega\} \tag{28}$$

where $\omega^3 = 1$, $\omega \neq 1$.

(vii) d_1: $R^{222}_{111} = k \neq 0$. There are 18 solutions but these solutions can be written compactly as

$$\{1, a, \lambda\omega' a, \omega, a, \pm\omega', \omega, \omega' a\} \tag{29}$$

where $\omega^3 = \omega'^3 = -1$ and $\lambda^2 = \pm 1$.

(viii) d_2: $R^{221}_{112} = k \neq 0$.

$$\{1, a, \pm a, 1, a, \pm 1, 1, 1\}. \tag{30}$$

(ix) d_4: $R^{211}_{122} = k \neq 0$.

$$\{1, a, \pm a, 1, a, \pm 1, 1, 1\}. \tag{31}$$

This completes the list of solutions possible under the diagonal ansatz with one non-zero off-diagonal element.

4.2. Solutions with two off-diagonal elements

There are $\binom{56}{2}$, i.e. 1540 possible positions. Again, most solutions are related to each other by the similarity transformation and discrete symmetries. In particular, there are two interesting cases to consider:

Case 1: The two non-zero elements are symmetrical about the diagonal.

Case 2: The two elements are symmetrical about the cross diagonal.

4.2.1. Case 1. There are altogether four different types of non-singular solutions. They correspond, modulo the symmetry transformations, to the cases when R_{111}^{112}, R_{111}^{121}, R_{112}^{122}, and R_{112}^{212} and their respective entries by reflection about the diagonal of the R-matrix are non-zero, for example, when R_{111}^{112} and R_{112}^{111} are non-zero off-diagonal elements, and so forth.

If we denote the diagonal entries of the solutions by

$$\{R_{111}^{111}, R_{112}^{112}, R_{121}^{121}, R_{122}^{122}, R_{211}^{211}, R_{212}^{212}, R_{221}^{221}, R_{222}^{222}\} \tag{32}$$

we can present the solutions as follows.

(i) When R_{111}^{112}, R_{112}^{111} are not zero. There are three distinct sets of solutions:

$$\{1, -1, a, \pm a, b, \pm b, c, d\} \tag{33}$$

with $R_{111}^{112} = (1-a^2)/k$ and $R_{112}^{111} = k$;

$$\{1, a, \pm(a+1), \pm(a+1), b, b, c, d\} \tag{34}$$

$$\{1, a, \pm(a+1), \mp(a+1), b, -b, c, d\} \tag{35}$$

with $R_{111}^{112} = a/k$ and $R_{112}^{111} = k$.

(ii) When R_{111}^{121}, R_{121}^{111} are not zero. There are two sets of solutions. They take the form

$$\left\{1, \pm 2, 1, a, \pm 2, \frac{4}{a}, a, b\right\} \tag{36}$$

with $R_{111}^{121} = 1/k$ or $-3/k$ and $R_{121}^{111} = k$, and

$$\left\{1, a+1, a, b, a+1, \frac{(a+1)^2}{b}, b, c\right\} \tag{37}$$

with $R_{111}^{121} = a/k$ and $R_{121}^{111} = k$.

(iii) When R_{112}^{122}, R_{122}^{112} are not zero. The solutions are

$$\{1, a, b, 1-a, 1, b, 1, c\} \tag{38}$$

with $R_{112}^{122} = a(1-a)/k$ and $R_{122}^{112} = k$, and

$$\left\{1, a, b, \frac{\pm\sqrt{b^2-4b}-b}{2}, \frac{-4b+2b^2\pm 2b^{\frac{3}{2}}\sqrt{b-4}}{4}, b, 1, c\right\} \tag{39}$$

with $R_{112}^{122} = a(1-a)/k$ and $R_{122}^{112} = k$.

(iv) When R_{112}^{212}, R_{212}^{112} are not zero. There is one solution

$$\left\{1, a, b, \frac{a+c}{b}, 1, c, b, \frac{a+c}{b}\right\} \tag{40}$$

with $R_{112}^{212} = ac/k$ and $R_{212}^{112} = k$.

These solutions seem to possess some regular patterns. If we label the vertices of a cube by {111},{112},{121}, and so forth as shown in figure 1, we make an interesting observation. Suppose we consider R_{ijk}^{lmn} as the non-zero element and look at the vertices on the cube corresponding to the indices $\{ijk\}$ and $\{lmn\}$, we see that a non-singular solution exists only for cases in which the vertices are connected by an edge of the cube.

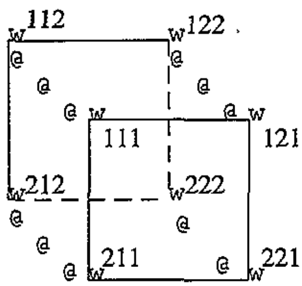

Figure 1. Labelling of cube.

4.2.2. *Case 2.* Non-singular solutions exist for the cases when R_{111}^{122}, R_{111}^{212}, R_{112}^{121}, R_{112}^{211}, R_{111}^{112}, R_{121}^{122} and R_{112}^{212} and their respective entries by reflection in the cross-diagonal of the R-matrix are not zero. These solutions seem less interesting than the previous cases. The off-diagonal elements are, in general, independent of the elements along the diagonal, except for case of solution (43).

Up to symmetries, there are seven basic solutions. Using the notations in the previous subsection 4.2.1, the solutions are:

(i)

$$\{a, b, a, b, a, b, a, b\} \tag{41}$$

with $R_{111}^{122} = c$ and $R_{211}^{222} = d$ and $a^2 = b^2$

(ii)

$$\{a, b, a, b, b, a, b, a\} \tag{42}$$

with $R_{111}^{212} = c$ and $R_{121}^{222} = d$ and $a^2 = b^2$

(iii)

$$\{a, b, b, -b, b, -b, -b, a\} \tag{43}$$

with $R_{112}^{121} = R_{212}^{221} = (a^2 - b^2)/2a$;

(iv)

$$\{a, a, a, -a, -a, a, a, a\} \tag{44}$$

with $R_{112}^{211} = b$ and $R_{122}^{221} = c$;

(v)

$$\{a, -a, a, a, b, b, -b, b\} \tag{45}$$

with $R_{111}^{112} = c$ and $R_{221}^{222} = d$;

(vi)

$$\{a, a, b, b, b, b, a, a\} \tag{46}$$

with $R_{121}^{122} = R_{211}^{212} = c$;

(vii)

$$\{a, b, b, a, a, c, c, a\} \tag{47}$$

with $R_{112}^{212} = R_{121}^{221} = d$.

4.2.3. Other cases. Besides the cases already mentioned, there are many solutions which are not related to the above solutions by any symmetry transformations mentioned in section 2.1. For example, in the case in which R_{111}^{112} is not zero and one other element systematically chosen from the other 27 possible positions in the upper triangle of the R-matrix is set as the non-zero element. In this case, we find solutions for cases in which

(i) $R_{111}^{221} \neq 0$: two possible solutions,
(ii) $R_{111}^{222} \neq 0$: two possible solutions,
(iii) $R_{121}^{122} \neq 0$: four possible solutions,
(iv) $R_{211}^{212} \neq 0$: four possible solutions,
(v) $R_{221}^{222} \neq 0$: eight possible solutions.

A typical solution from this list appears as

$$R = \begin{pmatrix} a & \frac{ac}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu b \end{pmatrix} \tag{48}$$

where λ, $\mu = \pm 1$.

4.3. Cross-diagonal ansatz with one or more non-zero elements

In comparison with the number of solutions which we can generate by looking at diagonal R-matrices with one or more off-diagonal elements, there are fewer solutions obtained from R-matrices with non-zero cross-diagonal elements and one or more elements off the cross-diagonal. By cross-diagonal ansatz, we refer to an R-matrix in which the elements

$$\{R_{111}^{222}, R_{112}^{221}, R_{121}^{212}, R_{122}^{211}, R_{211}^{122}, R_{212}^{121}, R_{221}^{112}, R_{222}^{111}\} \tag{49}$$

are not zero. We have shown earlier (see subsection 2.3) that such R-matrix will satisfy the FME when the parameters take on any value.

It is interesting to note that such an ansatz with one non-zero off cross-diagonal element does not yield any non-singular solution. Solutions exist only if the number of non-zero off cross-diagonal elements exceeds unity. We shall describe one such class of solutions: cross-diagonal ansatz with two non-zero elements placed symmetrically about the cross-diagonal.

4.3.1. Cross-diagonal ansatz with two non-zero elements. We shall list the cross-diagonal elements as

$$\{R_{111}^{222}, R_{112}^{221}, R_{121}^{212}, R_{122}^{211}. R_{211}^{122}, R_{212}^{121}, R_{221}^{112}, R_{222}^{111}\}.$$

For such an ansatz, we find that solutions exist essentially for two cases:

(i) $R_{111}^{122} = R_{211}^{222} = k$ The cross-diagonal elements are

$$\left\{a, b, \mu b, \lambda a, c, \frac{bc}{a}, \mu\frac{bc}{a}; \lambda c\right\}. \tag{50}$$

(ii) $R_{111}^{221} = R_{112}^{222} = k$. The cross-diagonal elements are

$$\left\{\lambda a, \lambda b, \mu c, \mu\frac{bc}{a}, c, \frac{bc}{a}, a, b\right\}. \tag{51}$$

4.4. Other solutions

As we increase the number of off-diagonal elements, we find increasing complexity in solving the 256 nonlinear 'quartic' equations. The number of over-determined consistency equations increases more rapidly than the increase in the number of unknowns. A typical result in which the four off-diagonal elements are symmetrical about both the diagonals is

$$R = \begin{pmatrix} a & a & 0 & 0 & 0 & 0 & 0 & 0 \\ b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 & d & d \end{pmatrix} \tag{52}$$

where $x^2 = (a+b)^2$ and $y^2 = (c+d)^2$.

One can easily check that Hietarinta's constant upper triangular solution of ZTE [4],

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{53}$$

satisfies the FME. We have also found a similar upper triangular solution which is not related to the above solution:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & p & 0 & 0 & q \\ 0 & 1 & 0 & 0 & 0 & p & -p & 0 \\ 0 & 0 & 1 & 0 & 0 & -p & p & 0 \\ 0 & 0 & 0 & 1 & p^2q^{-1} & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{54}$$

There is also an interesting 'bi-diagonal' solution analogous to Zamolodchikov's 'bi-diagonal' spectral dependent solution. This appears as

$$R = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & p \\ 0 & a & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & \pm b & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & a & r & 0 & 0 & p \\ 0 & 0 & 0 & q & b & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & \pm a & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 & a & 0 \\ qrp^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}. \tag{55}$$

5. Conclusion

As noted in Frenkel and Moore's original paper, the FME may be a good strategy to investigate Zamolodchikov's tetrahedron equation. The tetrahedron equation has so far admitted only one well known solution, which is the original spectral-dependent solution proposed by Zamolodchikov himself. On the other hand, there is a wealth of solutions, albeit spectral free, which we can generate from Frenkel and Moore's equation. By systematically increasing the number of unknowns in the R-matrix, we may be able to discover some symmetries which are inherent in both the tetrahedron equation and FME. Recently, Hu [21] has attempted to relate the FME to braid groups. More recently, Li and Hu [22] has shown that a given representation of the braid group induces a special kind of solution for the FME. They have also invoked symmetry transformations (5*b*) and (5*c*) in their solution. It would therefore seem that a systematic understanding of the FME will help us gain greater insights into higher dimensional generalization of the QYBE.

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